84.20 A formula for integrating inverse functions

The derivative of an inverse function is given in calculus textbooks (see, for example, [1]) by the formula

$$(f^{-1})'(y) = \frac{1}{f'(x)} \qquad \text{where } y = f(x).$$

Wouldn't it be useful also to have the integration counterpart? In fact, as far as we know, there is no calculus textbook that lists such a formula. Hence, the integration of inverse functions or expressions containing them can represent a problem to mathematics students, as it implies exceptional memory or the availability of integral tables. Here we deduce a simple theorem for integrating inverse functions based on a change of variable that does not require prior knowledge of their antiderivatives. Namely, if the function $y = f(x)$ has the integral $\int f(x)\,dx$, its inverse function $x = f^{-1}(y)$ can be easily integrated with the formula

$$\int f^{-1}(y)\,dy = xf(x) - \int f(x)\,dx.$$

The proof of this theorem is straightforward. Let the function $y = f(x)$ and the differential $dy = f'(x)\,dx$ be substituted in the integral

$$\int f^{-1}(y)\,dy = \int f^{-1}(f(x))f'(x)\,dx$$

to obtain

$$\int f^{-1}(y)\,dy = \int xf'(x)\,dx.$$

This expression can be integrated by parts to yield the required rule

$$\int f^{-1}(y)\,dy = xf(x) - \int f(x)\,dx.$$

To appreciate its power, let us take, for example $\int \cos^{-1} x\,dx$. Firstly, we introduce the change of variable $x = \cos(y)$ with $dx = -\sin(y)\,dy$, and then proceed:

$$\int \cos^{-1}(x)\,dx = y\cos(y) - \int \cos(y)\,dy$$

$$\int \cos^{-1}(x)\,dx = y\cos(y) - \sin(y) + C.$$

By changing back the variable, and recalling a familiar trigonometric identity, we obtain

$$\int \cos^{-1}(x)\,dx = x\cos^{-1}(x) - \sqrt{1 - x^2} + C.$$

This procedure is not new (see [2]), as mathematicians seem to invoke it unconsciously when they need to integrate an inverse function of unknown antiderivative (see, for example, [3]); it is essentially a subtle substitution

before integrating by parts; but because of this simple origin, it has not been formally stated, and thus newcomers have to discover it for themselves.

Finally, the present approach can be extended to integrate more complicated expressions containing inverse functions such as

$$\int F(y, f^{-1}(y))\,dy = F(f(x), x)f(x) - \int f(x)\frac{d}{dx}F(f(x), x)\,dx.$$

Acknowledgements

SS research has been funded by the JGH Award, ORS Award and CONICIT. We thank E. Crampin and D. McInerney for their comments.

References

1. Dale Varberg, Michael Sullivan and Edwin J. Purcell, *Calculus with analytical geometry*, (7th edn.), Prentice Hall, (1996).
2. R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, On the Lambert W function, *Adv. Comput. Math.* **5** (1996) pp. 329-359.
3. K. B. Ranger, A complex variable integration technique for the 2-dimensional Navier-Stokes equations, *Q. Applied Maths* **49** (1991) pp. 555-562.

S. SCHNELL

Centre for Mathematical Biology, Mathematical Institute

24-29 St Giles', Oxford OX1 3LB

e-mail: *schnell@maths.ox.ac.uk*

C. MENDOZA

Centro de Física, Instituto Venezolano de Investigaciones Científicas

(IVIC), PO Box 21827, Caracas 1020A, Venezuela

e-mail: *claudio@taquion.ivic.ve*